3.2 Nonlinear Models

POPULATION DYNAMICS

The **growth rate** is define by

$$\frac{dP/dt}{P}. \qquad (1)$$

In previous linear model we assumed that this grow rate is a constant K.

$$\left(dP/dt = KP \right).$$

We now assume that this rate is a function depending on P:

$$\frac{dP/dt}{P} = f(P), \text{ or } \frac{dP}{dt} = Pf(P) \qquad (2)$$

The equation (2) is called the **density-dependent hypothesis**. It is widely used models of animal population.

LOGISTC EQUATION

Assume that an environment is capable for sustaining no more than K of individuals in its population. The quantity K is called the **carrying capacity** of the environment.

Hence in function (2) we assume that $f(K) = 0$

and we simply let $f(0) = r$.

The simpliest function f satisfying the above two conditions is a linear function

$$f(P) = r - \left(\frac{r}{K}\right) P.$$

So equation (2) becomes

$$\frac{dP}{dt} = P\left(r - \frac{r}{K}P\right). \qquad (3)$$

Relabel the constant, (3) is the same as

(4) $$\frac{dP}{dt} = P(a - bP) \quad \left(a = r,\ b = \frac{r}{K}\right).$$

History: Eq. (4), for $a, b > 0$, was first studied by P. F. Verhulst (1804–1849) around 1840.

Equation (4) is known as the **logistic equation**.

SOLUTION OF LOGISTIC EQUATION

$$\frac{dP}{dt} = P(a - bP)$$

$$\Leftrightarrow \quad \frac{dP}{P(a-bP)} = dt$$

$$\Leftrightarrow \quad \left(\frac{1/a}{P} + \frac{b/a}{a - bP}\right) dP = dt$$

take integration

$$\Leftrightarrow \quad \frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| = t + c_1$$

$$\Leftrightarrow \quad \ln\left|\frac{P}{a-bP}\right| = at + ac_1$$

$$\Leftrightarrow \quad \frac{P}{a-bP} = c_2 e^{at}$$

Solve for P

$$P(t) = \frac{ac_2 e^{at}}{1+bc_2 e^{at}} = \frac{ac_2}{bc_2 + e^{-at}}$$

If $P(0) = P_0$ $\left(P_0 \neq \frac{a}{b}\right)$, we get

$$c_2 = P_0/(a-bP_0)$$

and

$$P(t) = \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}} . \qquad (5)$$

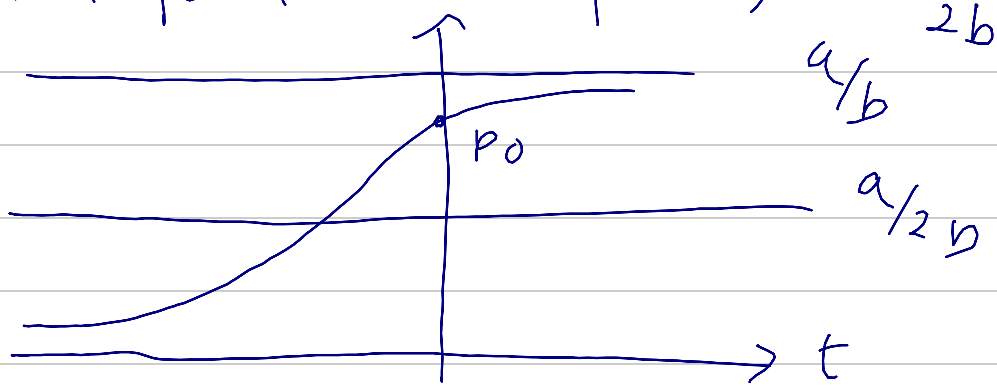
GRAPH OF P(t).

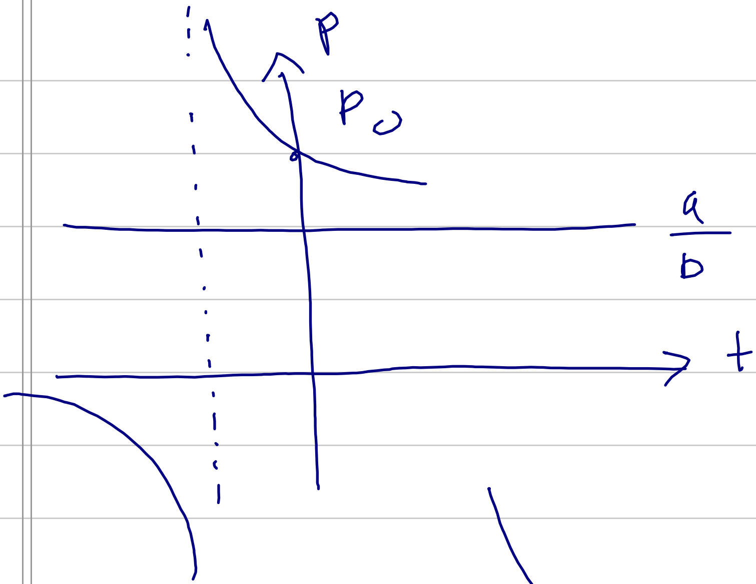
$$P(t) \to \frac{aP_0}{bP_0} = \frac{a}{b} \quad \text{as } t \to \infty.$$

$$P(t) \to 0 \quad \text{as } t \to 0.$$

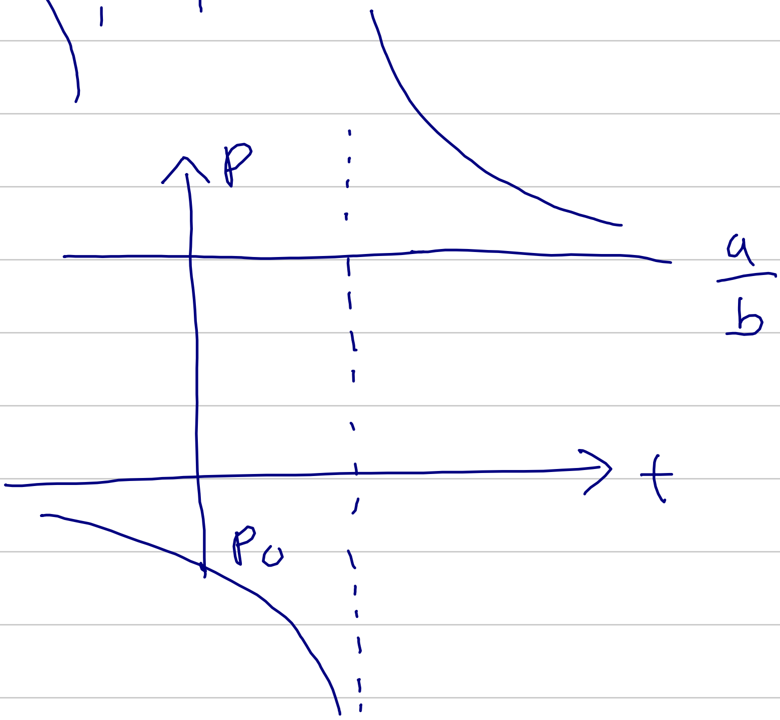
The reflection point (zero of P'') is $\frac{a}{2b}$

$0 < P_0 < a/b$





$P_0 > \frac{a}{b}$



$P_0 < 0$

However, P_0 should be positive, and P_0 should be less than $\frac{a}{b} = K$. Thus, we only interested in the case

$$0 < P_0 < \frac{a}{b}.$$

Example: Suppose a student carrying flu virus return to an isolated campus of 1000 students. If it is assume that the rate which the virus spreads is proportional not only to the number x of infected students but also the number of student not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

Solution: IVP

$$\frac{dx}{dt} = Kx(100-x), \quad x(0) = 1$$

$\Rightarrow$ $a = 1000K$ and $b = K$

$$x(t) = \frac{1000K}{K + 999K e^{-1000Kt}} = \frac{1000}{1 + 999 e^{-1000Kt}}.$$

$\boxed{x(4) = 50}$

$$50 = \frac{1000}{1 + 999 e^{-4000K}}$$

$\rightarrow$ $$-1000K = \frac{1}{4} \ln \frac{19}{999} = -0.9906.$$

Thus

$$x(t) = \frac{1000}{1 + 999 e^{-0.9906t}}$$

$\rightarrow$

$$x(6) = \frac{1000}{1 + 999 e^{-0.9906 \cdot 6}} = 276 \text{ students.}$$

MODIFICATIONS OF THE LOGISTIC EQ.

$$\frac{dP}{dt} = P(a - bP) \pm h \qquad (6)$$

could serve as models of population in a fishery where fish are harvested $(-h)$ or are restocked $(-h)$.

Or model with eq:

$$\frac{dP}{dt} = P(a-bP) - cP$$

where the harvest rate is propotional with P.

Or even

$$P' = P(a-bP) + ce^{-kP}$$

or

$$P' = P(a - b\ln P).$$

CHEMICAL REACTIONS

Suppose a grams of chemical A are combined with b grams of chemical B. If there are M parts of A and N parts of B formed in the compound and $X(t)$ is the number of grams of chemical C formed, then the remaining of A and B are, respectively

$$a - \frac{M}{M+N}X \quad \text{and} \quad b - \frac{N}{M+N}X.$$

The law of mass action states that the rate at which the two substances react is proportional to the product of the amounts of A and B that are remaining at time t:

$$\frac{dx}{dt} \sim \left(a - \frac{M}{M+N}x\right)\left(b - \frac{N}{M+N}x\right)$$

$$\sim \frac{MN}{(M+N)^2}\left(\frac{a(M+N)}{M} - x\right)\left(\frac{b(M+N)}{N} - x\right)$$

or

$$\boxed{\frac{dx}{dt} = k(\alpha - x)(\beta - x)} \qquad (9)$$

$(\alpha = a(M+N)/M, \quad \beta = b(M+N)/N)$.

Eq (9) is called the eq. of a second-order reaction.

Example. $A + B \to C$

For each gram of A, 4 grams of B is used. It is observed that 30gr of C is formed after 10mins. Determine the amount of C at time t. We have initially 50gr of A and 32gr of B.
How much C is present at 15mins?

Solution: $x(0) = 0, \quad x(10) = 30$

$M = 1, \quad N = 4.$

$$\frac{dx}{dt} \sim \left(50 - \frac{1}{5}x\right)\left(32 - \frac{4}{5}x\right)$$

$$(\Rightarrow) \quad dx/dt = k(250 - x)(40 - x)$$

$$\frac{dx}{(250-x)(40-x)} = k\,dt$$

Write $$\frac{1}{(250-x)(40-x)} = \frac{1}{210}\left(\frac{1}{40-x} - \frac{1}{250-x}\right)$$

$$\Leftrightarrow \quad -\frac{\frac{1}{210}}{250-x}\,dx + \frac{\frac{1}{210}}{40-x}\,dx = k\,dt$$

$$\Leftrightarrow \quad \ln\left|\frac{250-x}{40-x}\right| = 210\,kt + c_1$$

$$\Leftrightarrow \quad \left|\frac{250-x}{40-x}\right| = c_2\, e^{210kt}$$

$\boxed{t=0}$ $$c_2 = \frac{25}{4}$$

$\boxed{t=10}$ $$210\,k = \frac{1}{10}\ln\frac{88}{25} = 0.1258$$

Thus $$\left|\frac{250-x}{40-x}\right| = \frac{25}{4}e^{-0.1258t}$$

If $$\frac{250-x}{40-x} = \frac{25}{4}e^{-0.1258t},$$

then solve for x, we get

$$x(t) = 1000\,\frac{1-e^{-0.1258t}}{25-4e^{-0.1258t}}$$

$\Rightarrow$ $x(15) = 34.78$ gram. (and $x \to 40$ as $t \to \infty$).

Q: Why we can assume that

$$\left|\frac{250-x}{40-x}\right| = \frac{250-x}{40-x} ?$$